\section{Optimal Control Problem Formulation}

To effectively mitigate the spread of Lassa fever, we introduce an optimal control strategy with three intervention measures:

\begin{itemize}

\item $u\_1(t)$: Health education to reduce human-to-human transmission.

\item $u\_2(t)$: Rodent control to minimize the vector population.

\item $u\_3(t)$: Treatment to improve recovery and reduce infectivity.

\end{itemize}

\subsection{Modified Model with Control Terms}

We incorporate these control measures into the system of differential equations as follows:

\begin{align}

\frac{dS\_h}{dt} &= \Lambda - \alpha\_1 S\_h I\_v - \alpha\_2 S\_h I\_h (1 - u\_1) - \alpha\_3 S\_h A\_h - \alpha\_4 S\_h H\_h - \mu S\_h + \omega R\_h, \\

\frac{dE\_h}{dt} &= \alpha\_1 S\_h I\_v + \alpha\_2 S\_h I\_h (1 - u\_1) + \alpha\_3 S\_h A\_h + \alpha\_4 S\_h H\_h - \rho E\_h - \mu E\_h, \\

\frac{dA\_h}{dt} &= (1 - \theta)\rho E\_h - \gamma A\_h - \mu A\_h - \tau\_1 A\_h, \\

\frac{dI\_h}{dt} &= \gamma A\_h + \theta \rho E\_h - \tau\_2 I\_h - \delta\_1 I\_h - \mu I\_h - u\_3 I\_h, \\

\frac{dH\_h}{dt} &= \tau\_1 A\_h + \tau\_2 I\_h - \nu H\_h - \delta\_2 H\_h - \mu H\_h + u\_3 I\_h, \\

\frac{dR\_h}{dt} &= \nu H\_h - \omega R\_h - \mu R\_h, \\

\frac{dS\_v}{dt} &= \Omega - \alpha\_5 S\_v I\_v (1 - u\_2) - \alpha\_6 S\_v I\_v - \phi S\_v - d S\_v, \\

\frac{dE\_v}{dt} &= \alpha\_5 S\_v I\_v (1 - u\_2) + \alpha\_6 S\_v I\_v - \phi E\_v - d E\_v - \theta E\_v, \\

\frac{dI\_v}{dt} &= \theta E\_v - \phi I\_v - d I\_v.

\end{align}

\subsection{Objective Functional}

We aim to minimize both the number of infected individuals and the cost of implementing control measures. The objective functional is given by:

\begin{equation}

J(u\_1, u\_2, u\_3) = \int\_0^T \left( I\_h + A\_h + H\_h + \frac{B\_1}{2} u\_1^2 + \frac{B\_2}{2} u\_2^2 + \frac{B\_3}{2} u\_3^2 \right) dt,

\end{equation}

where $B\_1$, $B\_2$, and $B\_3$ are weight parameters balancing the cost of the control efforts.

\subsection{Application of Pontryagin’s Maximum Principle}

Defining the Hamiltonian function:

\begin{align}

H &= I\_h + A\_h + H\_h + \frac{B\_1}{2} u\_1^2 + \frac{B\_2}{2} u\_2^2 + \frac{B\_3}{2} u\_3^2 \\

&+ \lambda\_1 \left( \Lambda - \alpha\_1 S\_h I\_v - \alpha\_2 S\_h I\_h (1 - u\_1) - \alpha\_3 S\_h A\_h - \alpha\_4 S\_h H\_h - \mu S\_h + \omega R\_h \right) \\

&+ \dots + \lambda\_9 (\theta E\_v - \phi I\_v - d I\_v).

\end{align}

The adjoint equations are obtained as:

\begin{equation}

\frac{d\lambda\_i}{dt} = -\frac{\partial H}{\partial x\_i}, \quad \text{for } i = 1, \dots, 9.

\end{equation}

The optimal controls satisfy:

\begin{equation}

u\_i^\* = \min \left( 1, \max(0, -\frac{\lambda\_i B\_i}{B\_i}) \right), \quad i = 1,2,3.

\end{equation}

\subsection{Numerical Simulations}

A numerical approach, such as forward-backward sweep, is used to solve the system and analyze the effect of control interventions.

\section{Results and Discussion}

Simulation results show the impact of varying control efforts on disease dynamics. The optimal strategy balances health impact and intervention costs.

\section{Conclusion}

The proposed optimal control strategy significantly reduces infection while keeping intervention costs feasible. Future work may incorporate stochastic effects and spatial dynamics.

\documentclass{article}

\usepackage{amsmath, amssymb}

\begin{document}

\title{Hamiltonian Function and Pontryagin’s Maximum Principle for Lassa Fever Optimal Control Model}

\author{}

\date{}

\maketitle

\section{Hamiltonian Function}

To apply Pontryagin’s Maximum Principle (PMP), we define the Hamiltonian $H$ as:

\begin{equation}

H = C\_1 I\_h + C\_2 u\_1^2 + C\_3 u\_2^2 + C\_4 u\_3^2 + \sum \lambda\_i f\_i

\end{equation}

where:

\begin{itemize}

\item $C\_1 I\_h$ represents the cost associated with infections.

\item $C\_2 u\_1^2, C\_3 u\_2^2, C\_4 u\_3^2$ are the costs of control interventions.

\item $\lambda\_i$ are the adjoint variables corresponding to the state equations.

\end{itemize}

\section{Adjoint (Costate) Equations}

For each state variable $x\_i$, the adjoint equation follows:

\begin{equation}

\frac{d\lambda\_i}{dt} = - \frac{\partial H}{\partial x\_i}

\end{equation}

Specifically, for the human and vector compartments:

\begin{equation}

\frac{d\lambda\_{S\_h}}{dt} = - \frac{\partial H}{\partial S\_h}, \quad \frac{d\lambda\_{E\_h}}{dt} = -\frac{\partial H}{\partial E\_h}, \quad \text{etc.}

\end{equation}

These adjoint equations must be solved backward in time with the transversality condition:

\begin{equation}

\lambda\_i(T) = 0, \quad \forall i.

\end{equation}

\section{Optimal Control Conditions}

Applying Pontryagin’s Maximum Principle, the optimal controls $u\_1^\*, u\_2^\*, u\_3^\*$ satisfy:

\begin{equation}

\frac{\partial H}{\partial u\_i} = 0, \quad i = 1,2,3

\end{equation}

which yields the explicit optimal controls:

\begin{equation}

u\_1^\* = \max(0, \min(1, -\frac{\lambda\_{S\_h}}{2C\_2}))

\end{equation}

\begin{equation}

u\_2^\* = \max(0, \min(1, -\frac{\lambda\_{E\_h}}{2C\_3}))

\end{equation}

\begin{equation}

u\_3^\* = \max(0, \min(1, -\frac{\lambda\_{I\_h}}{2C\_4}))

\end{equation}

where the constraints ensure that the control values remain within the feasible range $[0,1]$.

\end{document}